

# EXISTENCE OF MAGNETIC COMPRESSIBLE FLUID STARS

PAUL FEDERBUSH, TAO LUO, JOEL SMOLLER<sup>1</sup>

**ABSTRACT.** The existence of magnetic star solutions which are axi-symmetric stationary solutions for the Euler-Poisson system of compressible fluids coupled to a magnetic field is proved in this paper by a variational method. Our method of proof consists of deriving an elliptic equation for the magnetic potential in cylindrical coordinates in  $\mathbb{R}^3$ , and obtaining the estimates of the Green's function for this elliptic equation by transforming it to 5-Laplacian.

## 1. INTRODUCTION

The purpose of this paper is to prove the existence of magnetic star solutions which are axi-symmetric stationary solutions for the following Euler-Poisson system of compressible fluids coupled to a magnetic field (cf. [6, 26, 27]):

$$(1.1) \quad \begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \\ (\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p(\rho) = -\rho \nabla \Psi + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}, \\ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \\ \nabla \cdot \mathbf{B} = 0, \\ \Delta \Psi = 4\pi G \rho. \end{cases}$$

Here  $\rho$ ,  $\mathbf{v} = (v_1, v_2, v_3)$ ,  $\mathbf{B}$ ,  $p(\rho)$  and  $\Phi$  denote the density, velocity, magnetic field, pressure and gravitational potential, respectively.  $G$  is the gravitational constant; we set it equal to 1 for simplicity. The gravitational potential is given by

$$(1.2) \quad \Phi(x) = - \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|} dy = -\rho * \frac{1}{|x|},$$

where  $*$  denotes convolution.

Throughout this paper, we assume that the pressure function  $p(\rho)$  satisfies the usual  $\gamma$ -law,

$$(1.3) \quad p(\rho) = \rho^\gamma, \quad \rho \geq 0,$$

for some constant  $\gamma > 1$ .

In this paper, we are interested in the stationary axi-symmetric solutions of (1.1) which represent an important class of equilibrium configurations. The stationary solutions ( $\mathbf{v} = \mathbf{0}$ ) satisfy the following system:

$$(1.4) \quad \begin{cases} \nabla p(\rho) = -\rho \nabla \Phi + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}, \\ \nabla \cdot \mathbf{B} = 0, \\ \Delta \Phi = 4\pi \rho. \end{cases}$$

There have been extensive studies on gaseous stars without taking magnetic effects into account, both for non-rotating and rotating stars; the reader may refer to [4, 1, 2, 3, 8, 9, 15, 5, 21, 17, 24] for the existence and properties of those solutions, and to [20, 18,

<sup>1</sup>Smoller was supported by the NSF, Grant no, DMS-1105189.

19, 22, 23, 25, 11, 12, 13, 14] for stability and instability (nonlinear or linear) in various settings. However, as far as we know, there have been no rigorous mathematical results on magnetic stars. The effects of magnetic fields arise in some physically interesting and important phenomena in astrophysics; eg. solar flares. As noted in [6]: “*The coupling between magnetic and thermomechanical degrees of freedom is observed in the solar flares (eruption phenomena in the coronal region of the Sun). During this spectacular event, a violent brightening is produced in the solar atmosphere where a huge amount of energy ( $\sim 10^{25}$  joules) is released in a matter of few minutes, and associated to a large coronal mass ejection. Magnetic reconnection is thought to be the mechanism responsible for this conversion of magnetic energy into heat and fluid motion.*” The aim of this paper is to give the first proof of the existence of stationary magnetic star solutions with prescribed total mass.

We prove our existence theorem via a variational technique as done in the non-magnetic case; see eg. [1, 5, 9, 22, 23, 25]. In these papers, the idea is to minimize an energy functional over a certain class  $W_M$  of  $\rho \in L^\gamma(\mathbb{R}^3)$ ,  $\gamma \geq 4/3$ , subject to a total mass constraint

$$\int_{\mathbb{R}^3} \rho(x) dx = M,$$

where  $M$  is a given positive number. The principal mathematical difficulty is that the energy functional is not of fixed sign. In our case where the Euler-Poisson equations are coupled to a magnetic field, the problem becomes more challenging.

The coupling to a magnetic field alone arises because stars seldom have a net charge ([20]), so the electric field vanishes. Thus in order to take electro-magnetic effects into account, a non-trivial current  $\mathbf{J}$  must be present. Since the current in a star is quite complicated (and not known even for the Sun), we take a special ansatz for  $\mathbf{J}$ . Proving the existence of a solution to the equations (1.4) with this ansatz demonstrates the consistency of our model.

There are two important inequalities needed in our existence results; namely, if  $F(\rho)$  denotes the energy functional defined on some class of functions  $W_M$ , then we need to prove

$$(i) \quad \inf_{\rho \in W_M} F(\rho) < 0,$$

and

$$(ii) \quad -\infty < \inf_{\rho \in W_M} F(\rho).$$

Inequality (i) shows that the gravitational energy dominates the other terms in the energy functional so that the star “holds together.” The second inequality implies that on any minimizing sequence, the energy functional is bounded from below.

In §2 we set up the problem in a convenient manner. In §3 we frame the problem variationally. In §4 we prove the main theorem. This states that if  $\gamma > 2$ , the energy functional has a minimizer in  $W_M$ .

Our method of proof consists of deriving an elliptic equation for the magnetic potential  $\psi$  in cylindrical coordinates,  $r = (x_1^2 + x_2^2)^{1/2}$ ,  $z = x_3$ ; namely

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = -4\pi\beta r^2\rho$$

where  $\rho \in W_M$  and  $\beta$  is a free parameter. To solve this equation we first transform  $\psi(x)$  to a certain function  $\chi(x)$ ,  $x \in \mathbb{R}^3$ . Then we extend  $\chi$  and  $\rho$  to functions  $\chi_e$  and  $\rho_e$  on  $\mathbb{R}^5$  which satisfy

$$\Delta_5 \chi_e = -4\pi\beta\rho_e,$$

where  $\Delta_5$  denotes the 5-Laplacian. This enables us to write  $\chi_e$  as a convolution of  $\rho_e$  with the Green's function for  $\Delta_5$ . Using Hölder's inequality together with Young's inequality, we can estimate  $\chi_e$  and thus  $\psi$  too. These estimates are used to prove (i) and (ii) if  $\gamma > 2$ .

In the appendix we extend our results to  $\gamma = 2$  for sufficiently small  $\beta$ . This result seems relevant for computing a “Chandrasekhar (mass) limit” of certain recently discovered white dwarf stars, cf. [7]. By further restricting the class  $W_M$ , we employ Riesz potentials ([10]), to extend our results to  $\gamma > 8/5$ . In a second appendix, we prove the non-existence of stationary spherically symmetric magnetic stars; cf. [17].

## 2. FORMULATION OF THE PROBLEM

We consider axi-symmetric solutions of (1.4). Thus if  $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ , let  $r = \sqrt{x_1^2 + x_2^2}$ ,  $z = x_3$ . The solutions we seek have the form

$$(2.1) \quad \begin{cases} \rho(x) = \rho(r, z), \Phi(x) = \Phi(r, z), \\ \mathbf{B}(x) = B^r(r, z)\mathbf{e}_r + B^\theta(r, z)\mathbf{e}_\theta + B^z(r, z)\mathbf{e}_3. \end{cases}$$

Here

$$(2.2) \quad \mathbf{e}_r = (x_1/r, x_2/r, 0)^T, \quad \mathbf{e}_\theta = (-x_2/r, x_1/r, 0)^T, \quad \mathbf{e}_3 = (0, 0, 1)^T,$$

so  $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_3\}$  is the standard orthogonal basis in cylindrical coordinates. In this case,

$$(2.3) \quad \mathbf{B} = \left(\frac{x_1}{r}B^r - \frac{x_2}{r}B^\theta\right)\mathbf{i} + \left(\frac{x_2}{r}B^r + \frac{x_1}{r}B^\theta\right)\mathbf{j} + B^z\mathbf{k},$$

and thus

$$(2.4) \quad \nabla \times \mathbf{B} = \left(\frac{x_2}{r}g - \frac{x_1}{r}\partial_z B^\theta\right)\mathbf{i} - \left(\frac{x_1}{r}g + \frac{x_2}{r}\partial_z B^\theta\right)\mathbf{j} + \left(\frac{1}{r}B^\theta + \partial_r B^\theta\right)\mathbf{k},$$

where

$$(2.5) \quad g = (\partial_r B^z - \partial_z B^r).$$

Furthermore,

$$(2.6) \quad \begin{aligned} & (\nabla \times \mathbf{B}) \times \mathbf{B} \\ &= \left[ -\frac{x_1}{r} \left( gB^z + \frac{(B^\theta)^2}{r} + B^\theta \partial_r B^\theta \right) - \frac{x_2}{r} \left( B^z \partial_z B^\theta + \frac{B^r B^\theta}{r} + B^r \partial_r B^\theta \right) \right] \mathbf{i} \\ &+ \left[ -\frac{x_2}{r} \left( gB^z + \frac{(B^\theta)^2}{r} + B^\theta \partial_r B^\theta \right) + \frac{x_1}{r} \left( B^z \partial_z B^\theta + \frac{B^r B^\theta}{r} + B^r \partial_r B^\theta \right) \right] \mathbf{j} \\ &+ (gB^r - B^\theta \partial_z B^\theta) \mathbf{k}. \end{aligned}$$

Also, it is easy to show

$$(2.7) \quad \nabla p(\rho) + \rho \nabla \Phi = \frac{x_1}{r}(\partial_r p(\rho) + \rho \partial_r \Phi)\mathbf{i} + \frac{x_2}{r}(\partial_r p(\rho) + \rho \partial_r \Phi)\mathbf{j} + (\partial_z p(\rho) + \rho \partial_z \Phi)\mathbf{k}.$$

Thus (1.4), (2.6) and (2.7) imply that

$$B^z \partial_z B^\theta + \frac{B^r B^\theta}{r} + B^r \partial_r B^\theta = 0.$$

If  $B^\theta = 0$ , then this is clearly satisfied. For simplicity, we consider the case

$$(2.8) \quad B^\theta = 0.$$

With this assumption, (2.4) reduces to

$$\nabla \times \mathbf{B} = g \left( \frac{x_2}{r} \mathbf{i} - \frac{x_1}{r} \mathbf{j} \right),$$

where  $g$  is given in (2.5). The magnetic current  $\mathbf{J}$  is defined by

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}.$$

If  $f = \frac{cg}{4\pi r}$ , where  $c$  is the speed of the light, our ansatz for the current density  $\mathbf{J}$  is the simplest one that supports a magnetic field; namely

$$(2.9) \quad \mathbf{J} = (x_2 \mathbf{i} - x_1 \mathbf{j}) f(r, z).$$

Conversely, we can show that if the current density  $\mathbf{J}$  takes the form of (2.9), then  $B^\theta = 0$ . Indeed, with  $\mathbf{B}$  given in (2.1), we have:

$$(2.10) \quad \begin{aligned} \frac{4\pi}{c} \mathbf{J} &= \nabla \times \mathbf{B} \\ &= \left( \frac{x_2}{r} (\partial_r B^z - \partial_z B^r) - \frac{x_1}{r} \partial_z B^\theta \right) \mathbf{i} \\ &\quad + \left( \frac{x_1}{r} (\partial_z B^r - \partial_r B^z) - \frac{x_2}{r} \partial_z B^\theta \right) \mathbf{j} \\ &\quad + \left( \frac{1}{r} B^\theta + \partial_r B^\theta \right) \mathbf{k}. \end{aligned}$$

Therefore, if  $\mathbf{J}$  takes the form of (2.9), we have:

$$(2.11) \quad B^\theta(r, z) = 0,$$

and in this case,

$$(2.12) \quad \frac{1}{r} (\partial_r B^z - \partial_z B^r) = \frac{4\pi}{c} f(r, z).$$

Next  $\nabla \cdot \mathbf{B} = 0$  implies:

$$(2.13) \quad \partial_r B^r + \frac{1}{r} B^r + \partial_z B^z = 0.$$

An easy calculation gives

$$(2.14) \quad (\nabla \times \mathbf{B}) \times \mathbf{B} = -\frac{x_1}{r} B^z (\partial_r B^z - \partial_z B^r) \mathbf{i} - \frac{x_2}{r} B^z (\partial_r B^z - \partial_z B^r) \mathbf{j} + B^r ((\partial_r B^z - \partial_z B^r) \mathbf{k}).$$

This together with (2.12) implies

$$(2.15) \quad \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{f}{c} (-x_1 B^z \mathbf{i} - x_2 B^z \mathbf{j} + r B^r \mathbf{k}).$$

Therefore, we have, by (1.4), (2.7) and (2.15),

$$(2.16) \quad \begin{cases} \partial_r p(\rho) + \rho \partial_r \Phi = -\frac{rf}{c} B^z, \\ \partial_z p(\rho) + \rho \partial_z \Phi = \frac{rf}{c} B^r. \end{cases}$$

Let

$$(2.17) \quad i(\rho) = \int_0^\rho \frac{p'(s)}{s} ds.$$

Then (2.16) implies

$$\begin{cases} \rho \partial_r (i(\rho) + \Phi) = -\frac{rf}{c} B^z, \\ \rho \partial_z (i(\rho) + \Phi) = \frac{rf}{c} B^r. \end{cases}$$

Now writing (2.13) in the form

$$(2.18) \quad \partial_r (r B^r) + \partial_z (r B^z) = 0$$

enables us to introduce a magnetic potential  $\psi$  such that

$$(2.19) \quad \partial_z \psi = r B^r, \quad \partial_r \psi = -r B^z.$$

In this paper, we consider the case when

$$(2.20) \quad \frac{f}{c\rho} = \text{const} =: \beta.$$

Then it follows from (2.18) and (2.19) that

$$\nabla(i(\rho) + \Phi - \beta\psi) = 0, \text{ whenever } \rho > 0.$$

Hence,

$$(2.21) \quad i(\rho) + \Phi - \beta\psi = \text{const} =: \lambda, \text{ in the region } \rho > 0,$$

where  $i(\rho)$  is given by (2.17), and  $\Phi$  is given by

$$(2.22) \quad \Phi(x) = - \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|} dy =: -\mathfrak{G}(\rho)(x).$$

Then solving (2.21) with the total mass constraint

$$(2.23) \quad \int_{\mathbb{R}^3} \rho(x) dx = M, \text{ for some given positive constant } M,$$

is the problem we consider in this paper.

### 3. VARIATIONAL FORMULATION

For  $p$  satisfying the  $\gamma$ -law, (1.3), let

$$(3.1) \quad A(\rho) = \frac{p(\rho)}{\gamma - 1}.$$

Then

$$(3.2) \quad i(\rho) = A'(\rho).$$

Also, the gravitational potential is given by (2.22), and we write  $\Phi = -\mathfrak{G}(\rho)$ . The magnetic potential  $\psi$  satisfies

$$(3.3) \quad \text{div}\left(\frac{1}{r^2} \nabla \psi\right) = -4\pi\beta\rho,$$

where  $r = \sqrt{x_1^2 + x_2^2}$ . Let  $G(x, y)$  for  $x, y \in \mathbb{R}^3$  be the Green's function for the operator  $\text{div}(\frac{1}{r^2} \nabla)$ , i. e.,

$$(3.4) \quad LG =: \text{div}\left(\frac{1}{r^2} \nabla G(x, y)\right) = \delta(x - y),$$

where  $\delta(x - y)$  is the Dirac mass. Since  $L$  is symmetric, we have

$$(3.5) \quad \langle L\psi, G \rangle = \langle \psi, LG \rangle = \langle \psi, \delta(x - y) \rangle = \psi(y),$$

where the inner product  $\langle \cdot, \cdot \rangle$  is taken in  $L^2$ . Thus we have the following integral representation for  $\psi$ , namely,

$$(3.6) \quad \psi(x) = \mathfrak{P}(\rho),$$

where the integral operator  $\mathfrak{P}$  is given by

$$(3.7) \quad \mathfrak{P}(\rho) = -4\pi\beta \int_{\mathbb{R}^3} G(x, y) \rho(y) dy.$$

Then, equation (2.21) can be written as

$$(3.8) \quad i(\rho) - \mathfrak{G}(\rho) - \beta\mathfrak{P}(\rho) = \lambda, \text{ whenever } \rho > 0.$$

In order to state our results, let's review the following results for the non-rotating non-magnetic star solutions: For  $0 < M < +\infty$ , define  $X_M$  by

$$(3.9) \quad X_M = \left\{ \rho : \mathbb{R}^3 \rightarrow \mathbb{R}, \rho \geq 0, a.e., \int_{\mathbb{R}^3} \rho(x) dx = M, \text{ and } \int_{\mathbb{R}^3} [A(\rho(x)) + \frac{1}{2}\rho(x)\mathfrak{G}(\rho)(x)] dx < +\infty \right\},$$

where  $A(\rho)$  is the function given in (3.1). For  $\rho \in X_M$ , we define the **energy functional**  $\tilde{F}$  for non-rotating non-magnetic stars by

$$(3.10) \quad \tilde{F}(\rho) = \int [A(\rho(x)) - \frac{1}{2}\rho(x)\mathfrak{G}(\rho(x))] dx.$$

We then have

**Theorem 3.1.** *Suppose that the pressure function  $p(\rho) = \rho^\gamma$  with  $\gamma > 4/3$ . Let  $\hat{\rho}$  be a minimizer of the energy functional  $\tilde{F}$  in  $X_M$  and let*

$$(3.11) \quad \Gamma_M = \{x \in \mathbb{R}^3 : \hat{\rho}(x) > 0\},$$

*then there exists a constant  $\lambda_N$  such that*

$$(3.12) \quad \begin{cases} A'(\hat{\rho}(x)) - B\hat{\rho}(x) = \lambda_M, & x \in \Gamma_M, \\ -\mathfrak{G}(\hat{\rho})(x) \geq \lambda_N, & x \in \mathbb{R}^3 - \Gamma_M. \end{cases}$$

The proof of this theorem is well-known, cf. [1] or [25].

*Remark 1.* We call the minimizer  $\hat{\rho}$  of the functional  $\tilde{F}$  in  $X_M$  a *non-rotating non-magnetic star solution*.

*Remark 2.* For  $\gamma > 4/3$ , it was proved in [17] that such a minimizer  $\hat{\rho}$  of the functional  $\tilde{F}$  in  $X_M$  exists and is actually radial and unique, and has compact support, i. e., for the given total mass  $M$ , there exists a unique constant  $R_M > 0$  such that

$$(3.13) \quad \begin{cases} \hat{\rho}(x) > 0, & \text{if } |x| < R_M, \\ \hat{\rho}(x) = 0, & \text{if } |x| \geq R_M. \end{cases}$$

In this case, we call  $R_M$  the *radius* of the non-rotating non-magnetic star solution with prescribed total mass  $M$ <sup>3</sup>.

Let  $W_M$  be the following function space

$$W_M = \left\{ \rho : \mathbb{R}^3 \rightarrow \mathbb{R}, \rho \text{ is axisymmetric}, \rho \geq 0, a.e., \rho \in L^1(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3), \int \rho(x) dx = M \right\},$$

and let  $W_M^*$  be defined by

$$(3.14) \quad W_M^* := \{ \rho \in W_M : \rho(r, z) = 0 \text{ for } r \geq \mathfrak{R} \},$$

for some positive constant  $\mathfrak{R} \geq R_M$  where  $R_M$  is the radius of the non-rotating non-magnetic star solution with prescribed total mass  $M$ , given in (3.13).

Define a functional  $F$  on  $W_M^*$  by

$$(3.15) \quad F(\rho) = \int \left( A(\rho) - \frac{1}{2}\rho\mathfrak{G}(\rho) - \frac{1}{2}\beta\rho\mathfrak{P}(\rho) \right) dx.$$

We now show that a minimizer of the functional  $F$  in  $W_M^*$  solves equation (3.8).

---

<sup>3</sup>In the appendix, we prove that radial solutions do not exist when magnetic fields are present.

**Theorem 3.2.** *Let  $\tilde{\rho}$  be a minimizer of the energy functional  $F$  in  $W_M^*$  and let*

$$(3.16) \quad \Gamma_M = \{x \in \mathbb{R}^3 : \tilde{\rho}(x) > 0\}.$$

*If  $\gamma > 6/5$ , then  $\tilde{\rho} \in C(\mathbb{R}^3) \cap C^1(\Gamma)$ . Moreover, there exists a constant  $\lambda_M$  such that*

$$(3.17) \quad A'(\tilde{\rho}(x)) - \mathfrak{G}(\tilde{\rho})(x) - \beta \mathfrak{P}(\tilde{\rho})(x) = \lambda_M, \quad x \in \Gamma.$$

*Proof.* We write  $F(\rho)$  in two parts:

$$(3.18) \quad F(\rho) = \tilde{F}(\rho) + I_2(\rho),$$

where

$$\tilde{F}(\rho) = \int_{\mathbb{R}^3} \left( A(\rho) - \frac{1}{2} \rho \mathfrak{G}(\rho) \right) dx,$$

and

$$I_2(\rho) = -\frac{1}{2} \beta \int_{\mathbb{R}^3} \rho \mathfrak{P}(\rho) dx.$$

For and  $\rho \in W_M^*$  and  $\rho + t\sigma \in W_M^*$  for  $t \in \mathbb{R}$  and  $\int_{\mathbb{R}^3} \sigma dx = 0$ , then using the same argument as in [1], we have

$$(3.19) \quad \lim_{t \rightarrow 0} \frac{\tilde{F}(\rho + t\sigma) - \tilde{F}(\rho)}{t} = \int (i(\rho) - \mathfrak{G}(\rho)) \sigma dx.$$

We calculate  $I_2(\rho + t\sigma) - I_2(\rho)$  as follows: by using (3.6) and (3.7), we obtain

$$\begin{aligned} & I_2(\rho + t\sigma) - I_2(\rho) \\ &= 2\pi\beta^2 \int \int G(x, y) \{(\rho + t\sigma)(x)(\rho + t\sigma)(y) - \rho(x)\rho(y)\} dx dy \\ (3.20) \quad &= 2\pi t\beta^2 \int \int G(x, y) (\sigma(x)\rho(y) + \rho(x)\sigma(y)) dx dy + 2\pi\beta^2 t^2 \int \int G(x, y) \sigma(x)\sigma(y) dx dy. \end{aligned}$$

Since  $G(x, y) = G(y, x)$ , we thus have

$$(3.21) \quad \lim_{t \rightarrow 0} \frac{I_2(\rho + t\sigma) - I_2(\rho)}{t} = 4\pi\beta^2 \int \int G(x, y) (\rho(y)\sigma(x)) dx dy = -\beta \int \mathfrak{P}(\rho)(x) \sigma(x) dx.$$

Therefore, by (3.19) and (3.21), we get

$$(3.22) \quad \lim_{t \rightarrow 0} \frac{F(\rho + t\sigma) - F(\rho)}{t} = \int (i(\rho) - \mathfrak{G}(\rho) - \beta \mathfrak{P}(\rho)) \sigma(x) dx,$$

for all  $\sigma$  such that  $\int \sigma(x) dx = 0$ . This, together with (3.19) proves the theorem, using a similar argument as in [1].  $\square$

The main theorem of this paper is the following:

**Theorem 3.3.** *Suppose that  $\gamma > 2$ . Then the following three statements hold:*

(1)

$$(3.23) \quad \inf_{W_M^*} F(\rho) < 0,$$

and

$$(3.24) \quad F(\rho) \geq C_1 \int_{\mathbb{R}^3} \rho^\gamma dx - C_2, \quad \rho \in W_M^*,$$

for some positive constants  $C_1$  and  $C_2$  independent of  $\rho$ .

(2) if  $\{\rho^i\} \subset W_M^*$  is a minimizing sequence for the functional  $F$ , then there exists a sequence of vertical shifts  $a_i \mathbf{e}_3$  ( $a_i \in \mathbb{R}$ ,  $\mathbf{e}_3 = (0, 0, 1)$ ), a subsequence of  $\{\rho^i\}$ , (still labeled  $\{\rho^i\}$ ), and a function  $\tilde{\rho} \in W_M^*$ , such that for any  $\epsilon > 0$  there exists  $R > 0$  with

$$(3.25) \quad \int_{a_i \mathbf{e}_3 + B_R(0)} \rho^i(x) dx \geq M - \epsilon, \quad i \in \mathbb{N},$$

and

$$(3.26) \quad T\rho^i(x) := \rho^i(x + a_i \mathbf{e}_3) \rightharpoonup \tilde{\rho}, \text{ weakly in } L^\gamma(\mathbb{R}^3), \text{ as } i \rightarrow \infty.$$

Moreover

(3)  $\tilde{\rho}$  is a minimizer of  $F$  in  $W_M^*$ .

Notice that (3.24) implies  $F$  is bounded from below. Thus any convergent minimizing sequence in  $W_M^*$  cannot tend to  $-\infty$ .

#### 4. PROOF OF THEOREM 3.3

In this section we prove Theorem 3.3. Statement (1) in Theorem 3.3 is crucial. With the aid of (1) in Theorem 3.3, (2) and (3) can be proved as in [22] and [23]. Therefore, the key is to prove (1) which is given by two lemmas.

First, we prove that the functional  $F(\rho)$  is bounded below on the set  $W_M^*$  if  $\gamma > 2$ .

**Lemma 4.1.** *Suppose that  $\gamma > 2$ . Then*

$$(4.1) \quad F(\rho) \geq C_1 \int_{\mathbb{R}^3} \rho^\gamma d_3 x - C_2, \quad \rho \in W_M^*,$$

for some positive constants  $C_1$  and  $C_2$  independent of  $\rho$ .

*Proof.* For  $\rho \in W_M^*$ , Let

$$(4.2) \quad F(\rho) = \tilde{F}(\rho) + \int_{\mathbb{R}^3} \rho(x) \mathfrak{P}(\rho)(x) d_3 x.$$

For simplicity of presentation, we set

$$(4.3) \quad 4\pi\beta = -1.$$

Let  $\psi = \mathfrak{P}(\rho)$ ; then  $\psi$  satisfies the following equation

$$(4.4) \quad \psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = r^2\rho.$$

It was shown in [22] or [23], for  $\rho \in W_M$  and  $\gamma > 4/3$ ,  $\tilde{F}$  satisfies the inequality

$$(4.5) \quad \tilde{F}(\rho) \geq c_1 \int (\rho(x))^\gamma d_3 x - c_2,$$

for some positive constants  $c_1$  and  $c_2$  independent of  $\rho$ . The main task is to estimate the term

$$(4.6) \quad Q = \int \rho(x) \psi(x) d_3 x.$$

To this end, we make the change of variable

$$(4.7) \quad \psi(r, z) = r^a \chi(r, z),$$



where  $a$  is a constant to be determined. We compute

$$\psi_{rr} - \frac{1}{r}\psi_r = r^a\chi_{rr} + (2a-1)r^{a-1}\chi_r + a(a-2)r^{a-2}\chi.$$

Taking  $a = 2$  gives

$$\psi_{rr} - \frac{1}{r}\psi_r = r^2\chi_{rr} + 3r\chi_r,$$

so using (4.4) we get

$$(4.8) \quad \chi_{rr} + \frac{3}{r}\chi_r + \chi_{zz} = \rho.$$

Noting that we are working with axi-symmetric functions, we recognize the left side of (4.8) to be related to the Laplacian of  $\chi$  in 5-dimensions. To make this precise, we must first extend our functions  $\rho$  and  $\chi$  from  $\mathbb{R}^3$  to  $\mathbb{R}^5$ . We do this as follows:

Let  $x_3 = z$ ,  $r = \sqrt{x_1^2 + x_2^2}$ ; then (4.8) becomes

$$(4.9) \quad \chi_{rr} + \frac{3}{r}\chi_r + \chi_{x_3x_3} = \rho(r, x_3), \quad \chi = \chi(r, x_3).$$

Now write

$$\rho(x_1, x_2, x_3) = f((x_1^2 + x_2^2)^{1/2}, x_3) = f(r, x_3),$$

and define the extension of  $\rho$  to  $\mathbb{R}^5$  by

$$\rho_e(x_1, x_2, x_3, x_4, x_5) = f((x_1^2 + x_2^2 + x_4^2 + x_5^2)^{1/2}, x_3) = f(R, x_3),$$

where

$$(4.10) \quad R = (x_1^2 + x_2^2 + x_4^2 + x_5^2)^{1/2}.$$

Similarly, writing

$$\chi(x_1, x_2, x_3) = g((x_1^2 + x_2^2)^{1/2}, x_3) = g(r, x_3),$$

we extend  $\chi$  to  $\mathbb{R}^5$  by defining

$$\chi_e(x_1, x_2, x_3, x_4, x_5) = g((x_1^2 + x_2^2 + x_4^2 + x_5^2)^{1/2}, x_3) = g(R, x_3).$$

Since (4.8) can be written as

$$g_{rr} + \frac{3}{r}g_r + g_{x_3x_3} = f(r, x_3),$$

it follows that

$$(4.11) \quad \frac{\partial^2 g}{\partial R^2} + \frac{3}{R} \frac{\partial g}{\partial R} + \frac{\partial^2 g}{\partial x_3^2} = f(R, x_3)$$

because the functions  $g(r, x_3)$  and  $g(R, x_3)$  are the same functions with different names for the first variable and  $f(r, x_3)$  is the same as  $f(R, x_3)$ , again with different names for the first variable. Thus, (4.8) gives

$$(\chi_e)_{RR} + \frac{3}{R}(\chi_e)_R + (\chi_e)_{x_3x_3} = \rho_e(R, x_3),$$

where  $R$  is given in (4.10). That is, the extended functions  $\chi_e$  and  $\rho_e$  satisfy

$$(4.12) \quad \Delta_5 \chi_e = \rho_e,$$

where  $\Delta_5$  denotes the Laplacian in  $\mathbb{R}^5$ . Now it is well-known ([10]) that the Green's function for  $\Delta_5$  is

$$(4.13) \quad G_5(x - y) = -\frac{1}{15\omega_5}|x - y|^{-3},$$

where  $\omega_5$  is the volume of the unit 5-ball. Thus, from (4.12) we obtain

$$(4.14) \quad \chi_e(x) = \int_{\mathbb{R}^5} G_5(x-y)\rho_e(y)d_5y = (G_5 * \rho_e)(x),$$

where  $*$  denotes the convolution operator.

We shall use (4.14) to study  $Q$ ; cf (4.6). Thus

$$(4.15) \quad Q = \int_{\mathbb{R}^3} \rho \psi d_3x = \int_{\mathbb{R}^3} \rho r^2 \chi d_3x = K \int_{\mathbb{R}^5} \rho_e \chi_e d_5x = K \int_{\mathbb{R}^5} \rho_e (G_5 * \rho_e) d_5x,$$

where  $K$  is the area of the unit 1-sphere divided by the area of the unit 3-sphere. Using Hölder's inequality we obtain

$$(4.16) \quad \left| \int_{\mathbb{R}^5} \rho_e (G_5 * \rho_e) d_5x \right| \leq \|\rho_e\|_s \|G_5 * \rho_e\|_t,$$

where

$$(4.17) \quad \frac{1}{s} + \frac{1}{t} = 1.$$

We would like to use Young's inequality ([16], P.19)

$$(4.18) \quad \|G_5 * \rho_e\|_t = \tilde{C} \| |x|^{-3} * \rho_e \|_t \leq C \tilde{C} \| |x|^{-3} \|_q \|\rho_e\|_s,$$

where  $C = C(q, s, t)$ , and

$$(4.19) \quad 1 + \frac{1}{t} = \frac{1}{q} + \frac{1}{s}.$$

To this end, we define the radial cut-off function  $\delta$  by

$$(4.20) \quad \delta(x-y) = \begin{cases} 1, & \text{if } |rad(x-y)| \geq 2\mathfrak{R}, \\ 0, & \text{otherwise.} \end{cases}$$

Here  $\mathfrak{R}$  is as in (3.14) and  $|rad(x-y)|$  is the distance of  $(x-y)$  from the  $x_3$  (or  $z$ )-axis. We now note that for  $\rho \in W_M^*$ , we may replace  $G_5$  by  $\delta G_5$  in (4.15), (4.16) and (4.18) and thus we need to study (from (4.18)),

$$(4.21) \quad \|\delta(x)|x|^{-3}\|_q,$$

where

$$(4.22) \quad \|\delta(x)|x|^{-3}\|_q^q = \int_{\mathbb{R}^5} \delta(x)|x|^{-3q} d_5x.$$

For this integral to be finite near  $x = 0$ , we need

$$(4.23) \quad q < 5/3.$$

From (4.17), (4.19) and (4.14), we obtain

$$(4.24) \quad s > \frac{10}{7}.$$

We will require  $\gamma \geq s$ . This is ensured for  $\gamma > 2$ . We next study the integral (4.22) at infinity.

We decompose the 5-vector  $x$  into its  $z$  and  $\bar{r}$  components:

$$x = a\bar{i}_z + \bar{b},$$

where  $\bar{i}_z$  is the unit vector in  $z$ -direction and  $\bar{b} = rad(x)$ . Writing  $d\Omega$  for the angular element, we have

$$\begin{aligned} \left\| \frac{\delta(x)}{|x|^3} \right\|_q^q &= \int_{\mathbb{R}^5} \frac{\delta(x)}{|x|^{3q}} d_5x = \int_{-\infty}^{+\infty} da \int d\Omega \int \frac{r^3}{(r^2 + a^2)^{3q/2}} dr \\ &\leq \int_{-1}^{+1} \int_0^{2\Re} r^3 dr \int d\Omega \frac{1}{(r^2 + a^2)^{3q/2}} da + 2 \int_1^{+\infty} \int_0^{2\Re} r^3 dr \int \frac{d\Omega}{a^{3q}} da. \end{aligned}$$

We will require

$$q \geq 1,$$

so both these expressions are finite. We now note from (4.15),

$$(4.25) \quad |Q| = K \int_{\mathbb{R}^5} \rho_e (G_5 * \rho_e) d_5x \leq CK \|\delta(x)|x|^{-3}\|_q \|\rho_e\|_s^2 \leq KC' \|\rho_e\|_s^2,$$

where  $C' = C\tilde{C}\|\delta(x)|x|^{-3}\|_q$  is a constant independent of  $\rho_e$ . This inequality, together with (4.2), (4.5) and (4.15), implies

$$(4.26) \quad F(\rho) \geq c_1 \int_{\mathbb{R}^3} \rho^\gamma(x) d_3x - C - C'K \|\rho_e\|_s^2.$$

We next estimate  $\|\rho_e\|_s^2$ . Before proceeding, we note that

$$(4.27) \quad \int_{\mathbb{R}^5} \rho_e d_5x = C_1 \int_{\mathbb{R}^3} r^2 \rho d_3x \leq C_1 \Re^2 M.$$

Then

$$\begin{aligned} \|\rho_e\|_s^s &= \int_{\rho_e < 1} \rho_e^s d_5x + \int_{\rho_e \geq 1} \rho_e^s d_5x \\ &\leq \int_{\rho_e < 1} \rho_e d_5x + \int_{\rho_e \geq 1} \rho_e^s d_5x. \end{aligned}$$

So

$$(4.28) \quad \|\rho_e\|_s^s \leq C_1 M \Re^2 + \int_{\rho_e \geq 1} \rho_e^2 d_5x \leq C_1 M \Re^2 + \int_{\rho_e \geq 1} \rho_e^\gamma d_5x,$$

and thus

$$(4.29) \quad |Q| \leq C' \left( C_1 M \Re^2 + \int_{\rho_e \geq 1} \rho_e^\gamma d_5x \right)^{2/s}.$$

At this point, we need the following elementary inequality

$$(4.30) \quad (x + y)^a \leq 2^a (x^a + y^a), \text{ for } x \geq 0, y \geq 0, a \geq 0.$$

Using (4.30), we get

$$(4.31) \quad \left( C_1 M \Re^2 + \int_{\rho_e \geq 1} \rho_e^\gamma d_5x \right)^{2/s} \leq (2C_1 M \Re^2)^{2/s} + (2 \int_{\rho_e \geq 1} \rho_e^\gamma d_5x)^{2/s}.$$

Applying the inequality ([10])

$$(4.32) \quad \alpha\beta \leq \epsilon\alpha^p + \epsilon^{-q/p}\beta^q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p > 1, q > 1,$$

gives for any  $\epsilon > 0$ ,

$$(2 \int_{\mathbb{R}^5} \rho_e^\gamma d_5x)^{2/s} \leq \epsilon \left( \int_{\rho_e \geq 1} \rho_e^\gamma d_5x \right)^{2p/s} + \epsilon^{-q/p} 2^{2q/s}.$$

Thus, assuming

$$(4.33) \quad s > 2,$$

and choosing  $p = \frac{s}{2} > 1$ , so  $q = (1 - \frac{1}{p})^{-1} = (1 - \frac{2}{s})^{-1} > 1$ , (4.31) implies

$$\begin{aligned} & \left( C_1 M \mathfrak{R}^2 + \int_{\rho_e \geq 1} \rho_e^\gamma d_5 x \right)^{2/s} \\ & \leq (2C_1 M \mathfrak{R}^2)^{2/s} + \epsilon^{-q/p} 2^{2q/s} + \epsilon \int_{\rho_e \geq 1} \rho_e^\gamma d_5 x. \end{aligned}$$

Then from (4.29)

$$|Q| \leq C'' + C' \epsilon \int_{\rho_e \geq 1} \rho_e^\gamma d_5 x,$$

where

$$C'' = C' \left[ (2C_1 M \mathfrak{R}^2)^{\frac{2}{s}} + \epsilon^{-2/p} 2^{2q/s} \right].$$

But

$$\begin{aligned} C' \epsilon \int_{\rho_e \geq 1} \rho_e^\gamma d_5 x & \leq C' \epsilon \int_{\mathbb{R}^5} \rho_e^\gamma d_5 x \\ & \leq C C' \epsilon \int_{\mathbb{R}^3} r^2 \rho^\gamma d_3 x \leq C C' \mathfrak{R}^2 \epsilon \int_{\mathbb{R}^3} \rho^\gamma d_3 x. \end{aligned}$$

Using this in (4.26), we obtain, by choosing  $\epsilon$  sufficiently small,

$$(4.34) \quad F(\rho) \geq \frac{c_1}{2} \int_{\mathbb{R}^3} \rho^\gamma d_3 x - C,$$

for some positive constants  $c_1$  and  $C$ . □

**Lemma 4.2.** *Suppose that  $\gamma > 4/3$ , then*

$$(4.35) \quad \inf_{W_M^*} F(\rho) < 0.$$

*Proof.* Let  $\hat{\rho}$  be the compactly supported solution for the non-rotating, non-magnetic star solution; cf (3.13). Then  $\hat{\rho} \in W_M^*$ . Moreover, by the argument in [22] or [23], we have

$$(4.36) \quad \tilde{F}(\hat{\rho}) < 0.$$

We use  $\psi$  to denote  $-4\pi\beta \int_{\mathbb{R}^3} G(x, y) \rho(y) dy = \mathfrak{P}(\rho)$ . Then

$$\operatorname{div}\left(\frac{1}{r^2} \nabla \psi\right) = -4\pi\beta \rho.$$

Thus, for any  $\rho \in W_M^*$ , we use the notation in Lemma 4.1, i.e.,

$$(4.37) \quad \psi(r, z) = r^2 \chi(r, z),$$

so that

$$(4.38) \quad \chi_{rr} + \frac{3}{r} \chi_r + \chi_{zz} = \rho.$$

Thus, re-inserting  $-4\pi\beta$ , (see (4.3)), (4.38) gives

$$(\chi_e)_{RR} + \frac{3}{R} (\chi_e)_R + (\chi_e)_{x_3 x_3} = -4\pi\beta \rho_e(R, x_3),$$

or equivalently

$$(4.39) \quad \Delta_5 \chi_e = -4\pi\beta \rho_e.$$

Therefore,

$$(4.40) \quad \chi_e = \int_{\mathbb{R}^5} G_5(x - y) (-4\pi\beta \rho_e) d_5 x,$$

where  $G_5(x - y)$  is the Green's function of  $\Delta_5$  given by (4.13). Notice that, for  $\rho \in W_{M^*}$ ,

$$(4.41) \quad \beta \int_{\mathbb{R}^3} \rho \psi d_3x = \beta \int_{\mathbb{R}^5} \rho_e(x) \chi_e(x) d_5x = -4\pi\beta^2 \int_{\mathbb{R}^5} \int_{\mathbb{R}^5} G_5(x - y) \rho_e(x) \rho_e(y) dx dy > 0,$$

due to (4.13). This implies, for any  $\rho \in W_{M^*}$ ,  $\beta \int \rho \psi(x) dx > 0$ . In particular,  $I_2(\hat{\rho}) =: -\frac{1}{2}\beta \int_{\mathbb{R}^3} \hat{\rho} \mathfrak{P}(\hat{\rho}) dx < 0$ . Since  $F(\hat{\rho}) = \tilde{F}(\hat{\rho}) + I_2(\hat{\rho})$ , we obtain  $F(\hat{\rho}) < 0$ . This proves (4.35).  $\square$

Lemmas 4.1 and 4.2 prove (1) in Theorem 3.3. With this, (2) and (3) in Theorem 3.3 can be proved as in [22] and [23].

## 5. APPENDIX

### Appendix A: Chandrasekhar Limit and the Case $\gamma = 2$

In a recent paper [7], there has been a discussion of the "Chandrasekhar limit" for magnetic white dwarf stars. White dwarfs avoid gravitational collapse via "electron degeneracy pressure" ([26]). This is a quantum mechanical effect resulting from the Pauli Exclusion Principle; namely, since electrons are fermions, no two electrons can be in the same state, and therefore occupy a band of energy levels. Compression of the electrons increases the number of electrons in a given volume and raises the maximum energy level in the occupied band. Thus the energy of the electrons increases, resulting in a pressure against the gravitational compression of matter into smaller volumes of space. The *Chandrasekhar limit* is the mass above which electron degeneracy pressure is insufficient to balance the stars own gravitational attraction.

In [7], the authors claim that "strongly magnetized white dwarfs not only can violate the Chandrasekhar mass limit significantly, but exhibit a different mass limit". In their analysis they consider a polytropic equation of state  $p = \rho^\gamma$  with  $\gamma = 2$ . Thus it is of some interest to extend Theorem 3.3 to the case  $\gamma = 2$ .

**Theorem A1:** Theorem 3.3 holds if  $\gamma = 2$  provided  $|\beta|$  is sufficiently small.

*Proof.* It suffices to show that (4.1) holds if  $\gamma = 2$  for small  $|\beta|$ .

As before, we define  $Q$  by

$$(5.1) \quad Q = -4\pi\beta \int_{\mathbb{R}^3} \rho \psi d_3x.$$

For  $s = 2$ , we have, as in (4.29),

$$\frac{|Q|}{4\pi|\beta|} \leq C' \left[ C_1 M \tilde{R}^2 + \int_{\rho_e \geq 1} \rho_e^2 d_5x \right],$$

so

$$(5.2) \quad \frac{|Q|}{4\pi|\beta|} \leq C' \left[ C_1 M \tilde{R}^2 + C \int_{\mathbb{R}^3} \rho^2 d_3x \right],$$

since

$$\int_{\rho_e \geq 1} \rho_e^2 d_5x \leq \int_{\mathbb{R}^5} \rho_e^2 d_5x \leq C \int_{\mathbb{R}^3} r^2 \rho^2 d_3x \leq C \mathfrak{R}^2 \int_{\mathbb{R}^3} \rho^2 d_3x.$$

Thus,

$$(5.3) \quad |Q| \leq 4\pi|\beta|(C' C_1 M \mathfrak{R}^2) + 4\pi|\beta| C \mathfrak{R}^2 \int_{\mathbb{R}^3} \rho^2 d_3x.$$

But

$$F(\rho) = \tilde{F}(\rho) - 4\pi\beta \int_{\mathbb{R}^3} \rho(x) \psi(x) dx$$

where

$$\tilde{F}(\rho) \geq c_1 \int_{\mathbb{R}^3} \rho^2 d_3 x - c_2.$$

Now choose  $|\beta|$  so small that

$$4\pi|\beta|C\mathfrak{R}^2 < \frac{c_1}{2}.$$

Then

$$F(\rho) \geq c_1 \int_{\mathbb{R}^3} \rho^2 d_3 x - c_2 - 4\pi|\beta|(CC'M\mathfrak{R}^2) - \frac{c_1}{2} \int_{\mathbb{R}^3} \rho^2 d_3 x,$$

which implies:

$$F(\rho) \geq \frac{c_1}{2} \int_{\mathbb{R}^3} \rho^2 d_3 x - c_2 - 4\pi|\beta|(CC'M\mathfrak{R}^2),$$

and this is (4.1).  $\square$

The last result was valid for  $\rho \in W_M^* := \{\rho \in W_M : \rho(r, z) = 0 \text{ for } r \geq \mathfrak{R}\}$ . If we consider  $\rho$  in a smaller class; namely,

$$(5.4) \quad W_M^{**} := \{\rho \in W_M : \rho(r, z) = 0 \text{ for } \sqrt{r^2 + z^2} \geq \mathfrak{R}\},$$

we can reduce  $\gamma$  below 2, with no restriction on  $\beta$ .

**Theorem A2:** If  $\rho \in W_M^{**}$ , then Theorem 3.3 holds for  $\gamma > 8/5$ .

*Proof.* As in (4.15), we have

$$Q = \int_{\mathbb{R}^3} \rho \psi d_3 x = K \int_{\mathbb{R}^5} \rho_e \chi_e d_5 x,$$

with

$$\chi_e(x) = -\frac{1}{15\omega_5} \int_{\mathbb{R}^5} \frac{\rho_e(y)}{|x-y|^3} d_5 y = -\frac{1}{15\omega_5} \int_{\Omega_5} \frac{\rho_e(y)}{|x-y|^3} d_5 y,$$

where

$$(5.5) \quad \Omega_5 =: \{x \in \mathbb{R}^5 : R = \sqrt{x_1^2 + x_2^2 + x_4^2 + x_5^2} \leq \mathfrak{R}, |x_3| \leq \mathfrak{R}\}.$$

By Hölder's inequality, we have

$$(5.6) \quad |Q| \leq K \|\rho_e\|_{2-\epsilon} \|\chi_e\|_{(2-\epsilon)/(1-\epsilon)}.$$

By the Reisz potential estimate (cf. [10] Lemma 7.12, p. 159), we obtain

$$(5.7) \quad \|\chi_e\|_{(2-\epsilon)/(1-\epsilon)} \leq C_p |\Omega_5|^{\mu-\delta} \|\rho_e\|_p, \quad p > \frac{5(2-\epsilon)}{9-7\epsilon},$$

for  $\mu = \frac{2}{5}$ ,  $\delta = \frac{1}{p} - \frac{1-\epsilon}{2-\epsilon}$ . Now Hölder's inequality states, if  $f \in L^q \cap L^r$  ( $1 \leq q < p < r < \infty$ ), then

$$\|f\|_p \leq \|f\|_q^a \|f\|_r^{1-a},$$

for

$$a = \frac{p^{-1} - r^{-1}}{q^{-1} - r^{-1}}.$$

Taking  $q = 1$ ,  $r = 2 - \epsilon$ ,

$$(5.8) \quad a = \frac{(2-\epsilon)/p - 1}{1 - \epsilon} < 1,$$

and using (5.6), (5.7), we obtain

$$(5.9) \quad |Q| \leq C |\Omega_5|^{\mu-\delta} \|\rho_e\|_{2-\epsilon} \|\rho_e\|_1^a \|\rho_e\|_{2-\epsilon}^{1-a} = C |\Omega_5|^{\mu-\delta} \|\rho_e\|_{2-\epsilon}^{2-a} \|\rho_e\|_1^a,$$

i.e.,

$$(5.10) \quad |Q| \leq C|\Omega_5|^{\mu-\delta} \left( \int_{\Omega_5} \rho_e d_5 x \right)^a \left( \int_{\Omega_5} \rho_e^{2-\epsilon} d_5 x \right)^{(2-a)/(2-\epsilon)}.$$

Suppose

$$(5.11) \quad \gamma > 2 - \epsilon.$$

Then writing (4.32) in the form

$$\alpha\beta \leq \lambda\alpha^p + \lambda^{-q/p}\beta^q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p > 1, \quad q > 1$$

with

$$\alpha = \rho_e^{2-\epsilon}, \quad \beta = 1, \quad p = \frac{2}{2-\epsilon}, \quad q = \frac{\gamma}{\gamma-(2-\epsilon)},$$

we obtain

$$(5.12) \quad \int_{\Omega_5} \rho_e^{2-\epsilon} d_5 x \leq \lambda \int_{\Omega_5} \rho_e^\gamma d_5 x + \lambda^{-(2-\epsilon)/(\gamma-(2-\epsilon))} |\Omega_5|,$$

for any positive constant  $\lambda$ . Therefore, it follows from (5.10) and (5.11) that

$$(5.13) \quad |Q| \leq C|\Omega_5|^{\mu-\delta} \left( \int_{\Omega_5} \rho_e d_5 x \right)^a \left( \lambda \int_{\Omega_5} \rho_e^\gamma d_5 x + \lambda^{-(2-\epsilon)/(\gamma-(2-\epsilon))} |\Omega_5| \right)^{(2-a)/(2-\epsilon)}.$$

We choose  $a = \epsilon$ , then by (5.8), we obtain

$$(5.14) \quad p = \frac{2-\epsilon}{1+\epsilon-\epsilon^2}.$$

Since it is required that  $p > \frac{5(2-\epsilon)}{9-7\epsilon}$  (see (5.7)), for  $p$  given by (5.14), this is equivalent to requiring

$$(5.15) \quad \epsilon < \frac{2}{5}.$$

Moreover, we require  $\gamma > 2 - \epsilon$ , (see (5.11)). So if  $\gamma > 8/5$ , (5.15) is ensured.

For  $a = \epsilon$ , we get from (5.13),

$$(5.16) \quad |Q| \leq C(M, \mathfrak{R}) \left( \lambda \int_{\mathbb{R}^3} \rho_e^\gamma d_3 x + \lambda^{-(2-\epsilon)/(\gamma-(2-\epsilon))} \right),$$

for some constant  $C(M, \mathfrak{R})$  depending on  $M$  and  $\mathfrak{R}$ , by noting that

$$\begin{aligned} \int_{\Omega_5} \rho_e d_5 x &= A \int_{-\mathfrak{R}}^{\mathfrak{R}} \int_0^{\mathfrak{R}} R^3 \rho_e(R, z) dR dz \leq A \mathfrak{R}^2 \int_{\mathbb{R}^3} \rho d_3 x = A \mathfrak{R}^2 M, \\ \int_{\Omega_5} \rho_e^\gamma d_5 x &\leq A \mathfrak{R}^2 \int_{\mathbb{R}^3} \rho^\gamma d_3 x, \end{aligned}$$

where  $A$  is a universal constant. By choosing  $\lambda$  sufficiently small, we get

$$F(\rho) \geq \frac{1}{2} \int_{\mathbb{R}^3} \frac{\rho^\gamma}{\gamma-1} d_3 x - C(M, \mathfrak{R}),$$

for  $\rho \in W_M^{**}$ . This finishes the proof, using the same argument as in the proof of Theorem 3.1.  $\square$

## Appendix B: Non-existence of Spherically Symmetric Magnetic Stars.

Radial magnetic stars cannot exist because there are no magnetic point charges. One can see this as a consequence of  $\nabla \cdot \mathbf{B} = 0$ . Namely, if  $\mathbf{B}$  is spherically symmetric, then

$$\mathbf{B} = B(r) \left( \frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r} \right), \quad r = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

Thus

$$\nabla \cdot \mathbf{B} = \sum_{i=1}^3 \partial_{x_i} \left( B(r) \frac{x_i}{r} \right),$$

and since  $\partial_{x_i} r = \frac{x_i}{r}$ ,  $i = 1, 2, 3$ , we have

$$\partial_{x_i} \left( B(r) \frac{x_i}{r} \right) = \partial_r B(r) \left( \frac{x_i}{r} \right)^2 + B(r) \left( \frac{1}{r} - \frac{x_i^2}{r^3} \right).$$

It follows that

$$0 = \nabla \cdot \mathbf{B} = \partial_r B(r) + \frac{2}{r} B(r),$$

so  $\partial_r(r^2 B(r)) = 0$  and thus

$$B(r) = \frac{c}{r^2}, \quad c = \text{const.}$$

If  $B$  is bounded as  $r \rightarrow 0+$ , then  $B(r) = 0$ .

If we allow the singularity at  $r = 0$  (i.e.,  $c \neq 0$ ), then if  $B_R$  is the  $R$ -ball in  $\mathbb{R}^3$ , the magnetic energy is

$$\frac{1}{8\pi} \int_{B_R} |\mathbf{B}|^2 d_3x = \frac{1}{8\pi} \int_0^R \frac{c^2}{r^4} 4\pi r^2 dr = \frac{c^2}{2} \int_0^R \frac{dr}{r^2} = \infty.$$

## REFERENCES

- [1] G. Auchmuty and R. Beals, Variational solutions of some nonlinear free boundary problems, Arch. Rat. Mech. Anal. 43, 255-271 (1971).
- [2] G. Auchmuty, The global branching of rotating stars, Arch. Rat. Mech. Anal. 114, 179-194 (1991)
- [3] L. Caffarelli and A. Friedman, The shape of axi-symmetric rotating fluid, J. Funct. Anal., 694, 109-142 (1980)
- [4] S. Chandrasekhar, Introduction to the Stellar Structure, *University of Chicago Press* (1939)
- [5] Chanillo, Sagun and Li, Yan Yan, On diameters of uniformly rotating stars. Comm. Math. Phys. 166, no. 2, 417-430 (1994)
- [6] Cox, J.P., Giuli, R.T.: Principles of stellar structure, I,II. New York: Gordon and Breach, 1968
- [7] Das, U., and Mukhopaddhay, B., New mass limit for white dwarf: super Chandrasekhar Type Ia supernova as a new standard candle, arXiv: 1301.5965 (astro-ph SR), 25 January, 2013.
- [8] A. Friedman & B. Turkington, Asymptotic estimates for an axi-symmetric rotating fluid. J. Func. Anal. 37, 136-163 (1980)
- [9] A. Friedman & B. Turkington, Existence and dimensions of a rotating white dwarf. J. Diff. Eqns. 42, 414-437 (1981)
- [10] Gilbarg, D. & Trudinger, N. S., Elliptic Partial Differential Equations of Second Order, 2nd Ed. Springer, (1983).
- [11] Y. Guo & G. Rein, Stable steady states in stellar dynamics, Arch. Rat. Mech. Anal. 147, 225-243 (1999)
- [12] Jang, Juhi, Nonlinear instability in gravitational Euler-Poisson systems for  $\gamma = \frac{6}{5}$ . Arch. Ration. Mech. Anal. 188 (2008), no. 2, 265-307
- [13] Juhi Jang, Nonlinear Instability Theory of Lane-Emden stars, arXiv:1211.2463.
- [14] Juhi Jang, Ian Tice, Instability theory of the Navier-Stokes-Poisson equations, arXiv:1105.5128
- [15] Li, Yan Yan, On uniformly rotating stars, Arch. Rat. Mech. Anal. 115, no. 4, 367-393 (1991)
- [16] E. H. Lieb & M. Loss, Analysis, American Mathematical Society, Providence, Rhode Island (1997).
- [17] E. H. Lieb & H. T. Yau, The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics. Comm. Math. Phys. 112, no. 1, 147-174 (1987)
- [18] N. R. Lebovitz, The virial tensor and its application to self-gravitating fluids, Astrophys. J. 134 (1961) 500-536.



- [19] N. R. Lebovitz & A. Lifschitz, Short-wavelength instabilities of Riemann ellipsoids, *Philos. Trans. Roy. Soc. London Ser. A* 354, no. 1709, 927–950 (1996).
- [20] S. S. Lin, Stability of gaseous stars in spherically symmetric motions, *SIAM J. Math. Anal.* 28, no. 3, 539–569 (1997)
- [21] T. Luo & J. Smoller, Rotating fluids with self-gravitation in bounded domains. *Arch. Rat. Mech. Anal.* 173, no. 3, 345–377 (2004)
- [22] T. Luo & J. Smoller, Nonlinear Dynamical Stability of Newtonian Rotating and Non-rotating White Dwarfs and Rotating Supermassive Stars, *Comm. Math. Phys.* 284, 425–457 (2008).
- [23] T. Luo & J. Smoller, Existence and Nonlinear Stability of Rotating Star Solutions of the Compressible Euler-Poisson Equations, *Arch. Rational Mech. & Anal.* 191, No.3, 447–496 (2009)
- [24] R. McCann, Stable rotating binary stars and fluid in a tube, *Houston Journal of Mathematics*, 32, no. 2, 603–631 (2006).
- [25] G. Rein, Non-linear stability of gaseous stars. *Arch. Rat. Mech. Anal.* 168, no. 2, 115–130 (2003)
- [26] S. H. Shapiro & S. A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars*, *WILEY-VCH*, (2004)
- [27] Zirin, H.: *Astrophysics of the Sun*. Cambridge: Cambridge University Press, 1988

Acknowledgement: JS would like to thank P. Smereka, S. Wu, and H. T. Yau for helpful conversations.

Paul Federbush,  
Department of Mathematics, University of Michigan, Ann Arbor  
pfed@umich.edu

Tao Luo,  
Department of Math. & Stat. Georgetown University,  
tl48@georgetown.edu

Joel Smoller  
Department of Mathematics, University of Michigan, Ann Arbor  
smoller@umich.edu